SPLIT-BY-NILPOTENT EXTENSIONS ALGEBRAS AND STRATIFYING SYSTEMS

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Abstract. Let Γ and Λ be artin algebras such that Γ is a split-by-nilpotent extension of Λ by a two sided ideal I of Γ. Consider the so-called change of rings functors $G := {}_{\Gamma}\Gamma_{\Lambda} \otimes_{\Lambda} -$ and $F := {}_{\Lambda}\Lambda_{\Gamma} \otimes_{\Gamma} -$. In this paper, we find the necessary and sufficient conditions under which a stratifying system $(\Theta, \leq)$ in modΛ can be lifted to a stratifying system $(G\Theta, \leq)$ in mod(Γ). Furthermore, by using the functors F and G, we study the relationship between their filtered categories of modules; and some connections with their corresponding standardly stratified algebras are stated (see Theorem 5.12, Theorem 5.15 and Theorem 5.18). Finally, a sufficient condition is given for stratifying systems in mod(Γ) in such a way that they can be restricted, through the functor F, to stratifying systems in mod(Λ).

1. Introduction.

Stratifying systems where introduced in [12, 20, 21, 27, 31] and developed in [16, 23, 24, 17, 26] with some applications, for example, in [10, 11, 13, 14, 15, 19, 22, 28].

Split-by-nilpotent extension algebras have been recently studied in various settings. For example, in almost split sequences [6], tilting modules [3] and quasi-tilted, laura, shod and weakly-shod algebras [5]. We study these extension algebras from the point of view of the theory of stratifying systems.

The paper is organized as follows. After a brief section of preliminaries, we devote Section 3 to the study of the functors $F = {}_{\Lambda}\Lambda_{\Gamma}\otimes_{\Gamma} : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$ and $G = {}_{\Gamma}\Gamma_{\Lambda} \otimes_{\Lambda} - : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$, where Γ is a split-by-nilpotent extension of Λ by a two sided ideal I of Γ.
In Section 4, we show that, for any $M \in \mathrm{mod}(\Lambda)$, the algebra $\mathrm{End}_{\Gamma}(GM)$ is always an split-by-nilpotent extension of $\mathrm{End}_{\Lambda}(M)$ by $\mathrm{Hom}_{\Lambda}(M, I \otimes M)$ (see Theorem 4.3).
The section 5 is the main section in the paper. We give necessary and sufficient conditions such that the image under G, of a stratifying system in $\mathrm{mod}(\Lambda)$, is a stratifying system in $\mathrm{mod}(\Gamma)$. Here, the main results are 5.12, 5.15, 5.18

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and 5.19. Finally, in Section 6, we give a sufficient condition (see 6.4) for stratifying systems in $\operatorname{mod}(\Gamma)$ in such a way that they can be restricted, through the functor F, to stratifying systems in $\operatorname{mod}(\Lambda)$.

2. Preliminaries.

Throughout this paper, the term algebra means *artin algebra* over a commutative artin ring R. For an algebra Λ, the category of finitely generated left Λ-modules is denoted by $\operatorname{mod}(\Lambda)$. We denote by $\operatorname{proj}(\Lambda)$ the full subcategory of $\operatorname{mod}(\Lambda)$ whose objects are the projective Λ-modules. Unless otherwise specified, all the modules are finitely generated. Furthermore, for any positive integer t, we set $[1,t] := \{1, 2, \cdots, t\}$.

Definition 2.1. [12, 20] *Let Λ be an algebra. A stratifying system $(\Theta, \leq)$, of size t in $\operatorname{mod}(\Lambda)$, consists of a family of indecomposable Λ-modules $\Theta = \{\Theta(i)\}_{i=1}^{t}$ and a linear order $\leq$ on the set $[1,t]$, satisfying the following two conditions.*

(a) $\operatorname{Hom}_\Lambda(\Theta(i), \Theta(j)) = 0$ *if* $i > j$.
(b) $\operatorname{Ext}^1_\Lambda(\Theta(i), \Theta(j)) = 0$ *if* $i \geq j$.

For a set Θ of Λ-modules, let $\mathfrak{F}(\Theta)$ be the subcategory of $\operatorname{mod}(\Lambda)$ consisting of the Λ-modules M having a Θ-filtration, that is, a sequence of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ such that each factor M_{i+1}/M_i is isomorphic to a module in Θ for all i.

Definition 2.2. [21] *Let Λ be an algebra. An Ext-projective stratifying system $(\Theta, \underline{Q}, \leq)$, of size t in $\operatorname{mod}(\Lambda)$, consists of two families of non-zero Λ-modules $\Theta = \{\Theta(i)\}_{i=1}^{t}$ and $\underline{Q} = \{Q(i)\}_{i=1}^{t}$, with $Q(i)$ indecomposable for all i, and a linear order $\leq$ on the set $[1,t]$, satisfying the following three conditions.*

(a) $\operatorname{Hom}_\Lambda(\Theta(i), \Theta(j)) = 0$ *if* $i > j$.
(b) *For each $i \in [1,t]$, there is an exact sequence*

$$\varepsilon_i : 0 \longrightarrow K(i) \longrightarrow Q(i) \xrightarrow{\beta_i} \Theta(i) \longrightarrow 0,$$

with $K(i) \in \mathfrak{F}(\{\Theta(j) : j > i\})$.
(c) *Q is Θ-projective. That is $\operatorname{Ext}^1_\Lambda(Q, \mathbf{\Theta}) = 0$, where $Q := \oplus_{i=1}^{t} Q(i)$ and $\mathbf{\Theta} := \oplus_{i=1}^{t} \Theta(i)$.*

Recall that (see [21, Corollary 2.13]) an Ext-projective stratifying system $(\Theta, \underline{Q}, \leq)$ gives the stratifying system $(\Theta, \leq)$. Furthermore, for a given a stratifying system $(\Theta, \leq)$, we know by [21, Corollary 2.15] that there is a unique, up to isomorphism, Ext-projective stratifying system $(\Theta, \underline{Q}, \leq)$. So, it is said that $(\Theta, \underline{Q}, \leq)$ is the Ext-projective stratifying system associated to the stratifying system $(\Theta, \leq)$. We also have the dual notion of the Ext-injective stratifying system $(\Theta, \underline{Y}, \leq)$ associated to the stratifying system $(\Theta, \leq)$ (see [12, 20, 21]).

The stratifying systems are related with the so-called standardly stratified algebras and so we introduce this notion. Let Λ be an algebra. For $M, N \in \operatorname{mod}(\Lambda)$, the **trace** $\operatorname{Tr}_M(N)$ of M in N, is the Λ-submodule of N generated by the images of all morphisms from M to N.

We next recall the definition (see [29, 9, 2, 7]) of the class of standard Λ-modules. Let n be the rank of the Grothendieck group $K_0(\Lambda)$. We fix a linear order $\leq$ on the set $[1,n]$ and a representative set ${}_\Lambda P = \{{}_\Lambda P(i) : i \in [1,n]\}$, containing one module of each iso-class of indecomposable projective Λ-modules. Observe, that the set ${}_\Lambda P$ determines the representative set ${}_\Lambda S = \{{}_\Lambda S(i)\}_{i=1}^{n}$ of simple Λ-modules, where ${}_\Lambda S(i) := {}_\Lambda P(i)/\operatorname{rad}({}_\Lambda P(i))$ for each i.

The set of **standard Λ-modules** is ${}_\Lambda\Delta = \{{}_\Lambda\Delta(i) : i \in [1,n]\}$, where ${}_\Lambda\Delta(i) = {}_\Lambda P(i)/\operatorname{Tr}_{\oplus_{j>i}\,{}_\Lambda P(j)}({}_\Lambda P(i))$. Then, ${}_\Lambda\Delta(i)$ is the largest factor module of ${}_\Lambda P(i)$ with composition factors only amongst ${}_\Lambda S(j)$ for $j \leq i$. The algebra Λ is said to be a **standardly stratified algebra**, with respect to the linear order $\leq$ on the set $[1,n]$, if $\operatorname{proj}(\Lambda) \subseteq \mathfrak{F}({}_\Lambda\Delta)$ (see [2, 7, 8]). In this case, it is also said that the pair $(\Lambda, \leq)$ is a standardly stratified algebra (or an ss-algebra for short).

Let Λ be an algebra and $\leq$ be a linear order on $[1,n]$, where $n = rk\,K_0(\Lambda)$. By [9], it follows that the pair $({}_\Lambda\Delta, \leq)$ is always a stratifying system (it is known as the **canonical stratifying system**). Furthermore, if $(\Lambda, \leq)$ is an ss-algebra, the representative set of the indecomposable projective Λ-modules ${}_\Lambda P = \{{}_\Lambda P(i)\}_{i=1}^{n}$ satisfies that the triple $({}_\Lambda\Delta, {}_\Lambda P, \leq)$ is the Ext-projective stratifying system associated to $({}_\Lambda\Delta, \leq)$.

The main connection between Ext-projective stratifying systems and the class of ss-algebras is given by the following result.

Theorem 2.3. [21] *Let $(\Theta, \underline{Q}, \leq)$ be an Ext-projective stratifying system of size t in $\operatorname{mod}(\Lambda)$, $\Gamma = \operatorname{End}_\Lambda(\overline{Q})^{op}$, $H = \operatorname{Hom}_\Lambda(Q, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$ and $L = Q \otimes_\Gamma - : \operatorname{mod}(\Gamma) \to \operatorname{mod}(\Lambda)$. Then, the following statements hold true.*

(a) *The family ${}_\Gamma P = \{H(Q(i)) : i \in [1,t]\}$ is a representative set of the indecomposable projective Γ-modules. In particular, Γ is a basic algebra and $rk\,K_0(\Gamma) = t$.*

(b) *$(\Gamma, \leq)$ is an ss-algebra, that is, $\operatorname{proj}(\Gamma) \subseteq \mathfrak{F}({}_\Gamma\Delta)$.*

(c) *The restriction $H|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}({}_\Gamma\Delta)$ is an exact equivalence of categories and $L|_{\mathfrak{F}({}_\Gamma\Delta)} : \mathfrak{F}({}_\Gamma\Delta) \to \mathfrak{F}(\Theta)$ is a quasi-inverse of $H|_{\mathfrak{F}(\Theta)}$.*

(d) *$H(\Theta(i)) \simeq {}_\Gamma\Delta(i)$, for all $i \in [1,t]$.*

(e) *$\operatorname{add}(Q) = \mathfrak{F}(\Theta) \cap {}^\perp\mathfrak{F}(\Theta)$, where $M \in {}^\perp\mathfrak{F}(\Theta)$ if and only if the restriction functor $\operatorname{Ext}^1_\Lambda(M, -)|_{\mathfrak{F}(\Theta)} = 0$.*

Another nice feature, for a stratifying system $(\Theta, \leq)$ of size t, is that an analogous of the Jordan-Holder's Theorem holds for the set of "relative simples" Θ in $\mathfrak{F}(\Theta)$. That is, for any $M \in \mathfrak{F}(\Theta)$ and all $i \in [1, t]$, the filtration multiplicity $[M : \Theta(i)]$ is well defined (see [21, Lemma 2.6 (c)]). Therefore, we have the so-called Θ-length $\ell_\Theta(M) := \sum_{i=1}^{t} [M : \Theta(i)]$ of M.

In what follows, we introduce some features about split-by-nilpotent extensions.

Definition 2.4. [3, 28] *Let Γ and Λ be algebras, and let I be a two-sided ideal of Γ. It is said that Γ is a* **split-by-nilpotent extension** *of Λ by I, if $I \subseteq \operatorname{rad}(\Gamma)$ and there is an exact sequence of abelian groups*

$$0 \longrightarrow I \longrightarrow \Gamma \xrightarrow{\pi} \Lambda \longrightarrow 0$$

such that π is an epimorphism of algebras and there is a morphism of algebras $\sigma : \Lambda \to \Gamma$ with $\pi\sigma = 1_\Lambda$.

In all that follows, we fix the two algebras Λ and Γ, and the two-sided ideal $I \trianglelefteq \Gamma$, such that Γ is a split-by-nilpotent extension of Λ by I. Observe that the morphisms of algebras $\sigma : \Lambda \to \Gamma$ and $\pi : \Gamma \to \Lambda$, induce in a natural way (change of rings), a bimodule structure on I, Γ and Λ. Furthermore, $\Gamma = \Lambda \oplus I$ as abelian groups, and the multiplicative structure of Γ can be seen as $\gamma_1\gamma_2 = (\lambda_1, i_1)(\lambda_2, i_2) = (\lambda_1\lambda_2, i_1\lambda_2 + \lambda_1 i_2 + i_1 i_2)$; in this case $\pi(\lambda, i) = \lambda$ and $\sigma(\lambda) = (\lambda, 0)$.

We also remark that in [1], the authors consider a quotient path algebra Γ and give sufficient conditions on a set of arrows $\mathcal{A}$ of the ordinary quiver of Γ, so that Γ is a split-by-nilpotent extension of $\Lambda := \Gamma/I$ by I, where I is the ideal of Γ generated by the set $\mathcal{A}$.

Remark 2.5. *The morphisms of algebras $\sigma : \Lambda \to \Gamma$ and $\pi : \Gamma \to \Lambda$ have the following properties.*

(a) *π is a morphism of $\Gamma - \Gamma$ bimodules. In particular, we have the exact sequence of $\Gamma - \Gamma$ bimodules*

$$0 \longrightarrow {}_\Gamma I_\Gamma \longrightarrow {}_\Gamma \Gamma_\Gamma \xrightarrow{\pi} {}_\Gamma \Lambda_\Gamma \longrightarrow 0.$$

(b) *π and σ are morphisms of $\Lambda - \Lambda$ bimodules. In particular*

$${}_\Lambda \Gamma_\Lambda = {}_\Lambda \Lambda_\Lambda \bigoplus {}_\Lambda I_\Lambda$$

as $\Lambda - \Lambda$ bimodules.

(c) ${}_\Lambda \Gamma_\Gamma \otimes_\Gamma {}_\Gamma \Lambda_\Lambda \simeq {}_\Lambda \Lambda_\Lambda \simeq {}_\Lambda \Lambda_\Gamma \otimes_\Gamma {}_\Gamma \Gamma_\Lambda$ *as $\Lambda - \Lambda$ bimodules.*

Lemma 2.6. *Let $M \in \operatorname{mod}(\Gamma)$ and consider ${}_\Lambda M$ as Λ-module given by the change of rings $\sigma : \Lambda \to \Gamma$. Then, there exist natural isomorphisms*

$$\operatorname{Hom}_\Gamma({}_\Gamma \Gamma_\Lambda, {}_\Gamma M) \simeq {}_\Lambda M \simeq \operatorname{Hom}_\Gamma({}_\Gamma \Lambda_\Lambda, {}_\Gamma M)$$

Proof. It is straightforward to see that the natural morphisms

$$\varphi_M : \mathrm{Hom}_\Gamma({}_\Gamma\Gamma_\Lambda, {}_\Gamma M) \to {}_\Lambda M \quad \text{and} \quad \psi_M : {}_\Lambda M \to \mathrm{Hom}_\Gamma({}_\Gamma\Lambda_\Lambda, {}_\Gamma M),$$

given by $\varphi_M(f) := f(1)$ y $\psi_M(m)(\lambda) = \sigma(\lambda)m$, are isomorphisms of Λ-modules such that $\psi_M^{-1}(f) = f(1)$ and $\varphi_M^{-1}(m)(\gamma) = \gamma m$. □

Lemma 2.7. $\mathrm{Hom}_\Lambda({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma) \simeq {}_\Gamma\Lambda_\Gamma$ *as* $\Gamma - \Gamma$ *bimodules.*

Proof. The morphism $\varphi : \mathrm{Hom}_\Lambda({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma) \to {}_\Gamma\Lambda_\Gamma$, given by $\varphi(f) := f(1)$ is an isomorphism of $\Gamma - \Gamma$ bimodules with inverse $\varphi^{-1}(\lambda)(x) = x\lambda$. □

3. The usual change of rings functors

Let Γ be a split-by-nilpotent extension of Λ by $I \trianglelefteq \Gamma$. We have the functors

$$\mathrm{mod}\,(\Gamma) \xrightarrow{F} \mathrm{mod}\,(\Lambda) \xrightarrow{G} \mathrm{mod}\,(\Gamma),$$

where $F := {}_\Lambda\Lambda_\Gamma \otimes_\Gamma -$ and $G := {}_\Gamma\Gamma_\Lambda \otimes_\Lambda -$. These functors are known as **change of rings functors**.

We also recall, that a functor $H : \mathcal{A} \to \mathcal{B}$, between additive categories, **reflects** zero objects if $H(A) = 0$ implies that $A = 0$ for any $A \in \mathcal{A}$. Furthermore, for a given class $\mathcal{X}$ of objects in $\mathcal{A}$, **the essential image** of the functor $H|_\mathcal{X} : \mathcal{X} \to \mathcal{B}$, which is denoted by $\mathrm{Im}\,(H|_\mathcal{X})$ or $H(\mathcal{X})$, is the full subcategory of $\mathcal{B}$ whose objects are all the objects $Z \in \mathcal{B}$ for which there is an object $X \in \mathcal{X}$ such that $Z \simeq H(X)$.

In the following lemma, we write down some well-known basic properties (see [3, 5, 25]), and for the convenience of the reader, it is included a proof.

Lemma 3.1. *For the change of rings functors F and G, the following statements hold true.*

(a) $FG \simeq 1_{\mathrm{mod}\,(\Lambda)}$.
(b) *The functors F and G are faithful. In particular, they reflect zero objects.*
(c) *For any $M \in \mathrm{mod}\,(\Lambda)$, the Γ-module $G(M)$ is indecomposable if and only if M is indecomposable.*
(d) *For any $N \in \mathrm{mod}\,(\Gamma)$, if the Λ-module $F(N)$ is indecomposable then N is indecomposable.*

Proof. (a) It follows from 2.5 (c).

(b) The fact that G is faithful follows from (a). Let us prove that F is also faithful. Indeed, for any $M \in \mathrm{mod}\,(\Gamma)$, it can be seen easily that $\varphi_M : {}_\Lambda\Lambda_\Gamma \otimes_\Gamma M \to {}_\Lambda M$, given by $\varphi_M(\lambda \otimes m) := \sigma(\lambda)m$, is an isomorphism of Λ-modules, where ${}_\Lambda M$ has the structure of Λ-module given by the change of rings $\sigma : \Lambda \to \Gamma$. Furthermore, for any $f : M \to N$ in $\mathrm{mod}\,(\Gamma)$, we have the

following commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "\varphi_M"'] & F(N) \arrow[d, "\varphi_N"] \\ {}_\Lambda M \arrow[r, "f"'] & {}_\Lambda N. \end{tikzcd}$$

Thus, if $F(f)=0$ then $f=0$; proving that F is faithful.

(c) Let $M \in \operatorname{mod}(\Lambda)$ be such that $G(M)$ is indecomposable. In particular, $M \neq 0$. If $M = M_1 \oplus M_2$ then $G(M) = G(M_1) \oplus G(M_2)$; and since $G(M)$ is indecomposable, we have that $G(M_1) = 0$ or $G(M_2) = 0$. Thus, by (b), it follows that $M_1 = 0$ or $M_2 = 0$; proving that M is indecomposable.
Let $M \in \operatorname{mod}(\Lambda)$ be an indecomposable Λ-module. So $M \neq 0$ and by (b) $G(M) \neq 0$. If $G(M) = N_1 \oplus N_2$ then by (a) $M = F(N_1) \oplus F(N_2)$. Therefore, using that M is indecomposable, it follows that $F(N_1) = 0$ or $F(N_2) = 0$. Thus, by (b), we have that $N_1 = 0$ or $N_2 = 0$; and so $G(M)$ is indecomposable.

(d) As in (c), the item (d) follows from the fact that F reflects zero objects.
□

Lemma 3.2. *Let I_Λ be a projective Λ-module. Then, the following statements hold true.*

(a) *Γ_Λ is a projective Λ-module and* $\operatorname{Tor}_1^\Gamma(\Lambda_\Gamma, -)|_{\operatorname{Im}(G)} = 0$.

(b) *For all $n \geq 0$ and any $X, Y \in \operatorname{mod}(\Lambda)$, we have that*

$$\operatorname{Ext}^n_\Gamma(G(X), G(Y)) \simeq \operatorname{Ext}^n_\Lambda(X, Y) \oplus \operatorname{Ext}^n_\Lambda(X, I \otimes_\Lambda Y).$$

Proof. (a) Since $\Gamma_\Lambda = \Lambda_\Lambda \oplus I_\Lambda$ (see 2.5 (b)), it follows that Γ_Λ is a projective Λ-module, and hence $\operatorname{Tor}_1^\Lambda(\Gamma_\Lambda, -) = 0$. In particular, we get that G is an exact functor.

Let $X \in \operatorname{Im}(G)$. Then there is an isomorphism $f : G(M) \to X$ for some $M \in \operatorname{mod}(\Lambda)$. Consider an exact sequence ending at M, that is, $\eta \;:\; 0 \to K \to P \overset{h}{\to} M \to 0$. Since G is an exact functor, we get the exact sequence $\eta' \;:\; 0 \to G(K) \to G(P) \overset{fG(h)}{\to} X \to 0$. Thus, by applying the functor F to η' and using 3.1 (a), we get an exact and commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tor}_1^\Gamma(\Lambda_\Gamma, X) \arrow[r] & FG(K) \arrow[r] \arrow[d] & FG(P) \arrow[r] \arrow[d] & FG(X) \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & K \arrow[r] & P \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

where the vertical arrows are isomorphisms; proving that $\operatorname{Tor}_1^\Gamma(\Lambda_\Gamma, X) = 0$.

(b) Since Γ_Λ is projective, by [30, Exercise 9.21], we get that

$$\mathrm{Ext}^n_\Gamma(G(X), G(Y)) \simeq \mathrm{Ext}^n_\Lambda(X, \mathrm{Hom}_\Gamma({}_\Gamma\Gamma_\Lambda, G(Y))).$$

Thus, the result follows from 2.5 (b) and 2.6. $\square$

Theorem 3.3. *For the change of rings functors F and G, the following statements hold true.*

(a) *The restriction functor $F|_{\mathrm{Im}\,(G)} : \mathrm{Im}\,(G) \to \mathrm{mod}\,(\Lambda)$ is an equivalence and $G : \mathrm{mod}\,(\Lambda) \to \mathrm{Im}\,(G)$ is a quasi-inverse. Moreover, if I_Λ is projective, then $F|_{\mathrm{Im}\,(G)}$ and G are exact functors.*

(b) $\mathrm{add}\,(GM) \subseteq \mathrm{Im}\,(G)$ *for any $M \in \mathrm{mod}\,(\Lambda)$. Thus, the restriction functor $F|_{\mathrm{add}\,(GM)} : \mathrm{add}\,(GM) \to \mathrm{add}\,(M)$ is an equivalence and $G|_{\mathrm{add}\,(M)} : \mathrm{add}\,(M) \to \mathrm{add}\,(GM)$ is a quasi-inverse.*

Proof. (a) Let $\varepsilon : FG \to 1_{\mathrm{mod}\,(\Lambda)}$ be the isomorphism of functors given in 3.1 (a). Firstly, we assert that $F|_{\mathrm{Im}\,(G)}$ is full. Indeed, let $X, Y \in \mathrm{Im}\,(G)$. Then there are isomorphisms $\alpha_1 : X \to G(M)$ and $\alpha_2 : Y \to G(N)$ for some $M, N \in \mathrm{mod}\,(\Lambda)$. We need to show that $F : \mathrm{Hom}_\Gamma(X, Y) \to \mathrm{Hom}_\Lambda(FX, FY)$ is surjective. For any $f \in \mathrm{Hom}_\Lambda(FX, FY)$, set $\overline{f} := F(\alpha_2) f F(\alpha_1^{-1})$, $f' := \varepsilon_N \overline{f} \varepsilon_M^{-1}$ and $h := \alpha_2^{-1} G(f') \alpha_1$. So an straightforward calculation gives us that $F(h) = f$, proving that $F|_{\mathrm{Im}\,(G)}$ is full. Observe, that $F|_{\mathrm{Im}\,(G)}$ is dense since $FG \simeq 1_{\mathrm{mod}\,(\Lambda)}$. Moreover, by 3.1 (b) it follows that $F|_{\mathrm{Im}\,(G)}$ is an equivalence. Furthermore, since $FG \simeq 1_{\mathrm{mod}\,(\Lambda)}$, we conclude that $G : \mathrm{mod}\,(\Lambda) \to \mathrm{Im}\,(G)$ is a quasi-inverse of $F|_{\mathrm{Im}\,(G)}$.

Finally, assume that I_Λ is projective. Then, by 3.2 (a), we get that $F|_{\mathrm{Im}\,(G)}$ and G are exact functors.

(b) Let $M \in \mathrm{mod}\,(\Lambda)$ and let $X \in \mathrm{add}\,(GM)$. Then, there exists $Z \in \mathrm{mod}\,(\Gamma)$ such that $X \oplus Z = GM^m$, for some m. Since G is an additive functor, we may assume that X is indecomposable. Thus, from the Krull-Remak-Schmidt Theorem and 3.1 (c), we get that $X \simeq GM'$ for some indecomposable direct summand M' of M and thus $X \in \mathrm{Im}\,(G)$. $\square$

Corollary 3.4. *The restriction functor $F|_{\mathrm{proj}\,(\Gamma)} : \mathrm{proj}\,(\Gamma) \to \mathrm{proj}\,(\Lambda)$ is an equivalence and $G|_{\mathrm{proj}\,(\Lambda)} : \mathrm{proj}\,(\Lambda) \to \mathrm{proj}\,(\Gamma)$ is a quasi-inverse.*

Proof. It follows from 3.3 (b) since $G({}_\Lambda\Lambda) \simeq {}_\Gamma\Gamma$ and $\mathrm{add}\,({}_\Gamma\Gamma) = \mathrm{proj}\,(\Gamma)$. $\square$

4. The functor G and split-by-nilpotent extensions

We recall that the term algebra means artin R-algebra over a commutative artinian ring R and Γ is a split-by-nilpotent extension of Λ by I. Consider the change of rings functor $G := {}_\Gamma\Gamma_\Lambda \otimes_\Lambda - : \mathrm{mod}\,(\Lambda) \to \mathrm{mod}\,(\Gamma)$. Recall that ${}_\Lambda\Gamma_\Lambda = {}_\Lambda\Lambda_\Lambda \bigoplus {}_\Lambda I_\Lambda$ as bimodules. Hence $\Gamma \otimes_\Lambda N = (\Lambda \otimes_\Lambda N) \oplus (I \otimes_\Lambda N)$ as Λ-modules. For the sake of simplicity, we some times write (M, N) instead of $\mathrm{Hom}_\Lambda(M, N)$.

For each pair $M, N \in \operatorname{mod}(\Lambda)$, we consider the isomorphism $\delta = \delta_{M,N}$ of R-modules

$$\operatorname{Hom}_\Lambda(M,N) \oplus \operatorname{Hom}_\Lambda(M, I \otimes N) \xrightarrow{\delta} \operatorname{Hom}_\Gamma(GM, GN),$$

which is obtained as the composition $\delta := \tau \circ \operatorname{Hom}_\Lambda(M, \varphi^{-1}) \circ \nu$ of the following isomorphism of R-modules:

(a) $\nu : (M,N) \oplus (M, I \otimes N) \to (M, \Lambda \otimes N) \oplus (M, I \otimes N) = (M, \Gamma \otimes N)$, where $\nu(f,g) := (\overline{f}, g)$ and $\overline{f}(m) := 1 \otimes f(m)$ for all $m \in M$.

(b) $\operatorname{Hom}_\Lambda(M, \varphi^{-1}) : (M, \Gamma \otimes N) \to (M, \operatorname{Hom}_\Gamma(\Gamma, GN))$, where

$$\varphi^{-1}(\gamma_1 \otimes n)(\gamma_2) = \gamma_2(\gamma_1 \otimes n) = \gamma_2\gamma_1 \otimes n$$

as can be seen from the proof of 2.6.

(c) $\tau : (M, \operatorname{Hom}_\Gamma(\Gamma, GN)) \to \operatorname{Hom}_\Gamma(\Gamma \otimes M, GN) = \operatorname{Hom}_\Gamma(GM, GN)$, where $\tau(\alpha)(\gamma \otimes m) = \alpha(m)(\gamma)$.

Proposition 4.1. *$\delta = (G, L) : (M,N) \oplus (M, I \otimes N) \to \operatorname{Hom}_\Gamma(GM, GN)$ as a matrix, where $L(g)(\gamma \otimes m) = \gamma g(m)$. Furthermore $L : \operatorname{Hom}_\Lambda(M, I \otimes N) \to \operatorname{Hom}_\Gamma(GM, GN)$ is a monomorphism.*

Proof. Consider the natural inclusion $i_1 : (M,N) \to (M,N) \oplus (M, I \otimes N)$. We assert that $\delta i_1 = G$. Indeed, let $f \in (M,N)$, $\gamma \in \Gamma$ and $m \in M$. So we have $\delta i_1(f)(\gamma \otimes m) = \tau(\varphi^{-1}\overline{f})(\gamma \otimes m) = (\varphi^{-1}\overline{f}(m))(\gamma) = \gamma(1 \otimes f(m)) = \gamma \otimes f(m) = G(f)(\gamma \otimes m)$.

Let $i_2 : (M, I \otimes N) \to (M,N) \oplus (M, I \otimes N)$ be the natural inclusion. We check that $\delta i_2 = L$. Indeed, let $g \in (M, I \otimes N)$, $\gamma \in \Gamma$ and $m \in M$. So we have $\delta i_2(g)(\gamma \otimes m) = (\tau\varphi^{-1}(0,g))(\gamma \otimes m) = \varphi^{-1}(0,g)(m)(\gamma) = \gamma g(m)$.

Finally, let $y \in \operatorname{Hom}_\Lambda(M, I \otimes N)$ be such that $L(y) = 0$. Then $\delta \begin{pmatrix} 0 \\ y \end{pmatrix} = G(0) + L(y) = 0$ and since δ is an isomorphism, it follows that $y = 0$; proving that L is a monomorphism. □

Corollary 4.2. *Let $M, N \in \operatorname{mod}(\Lambda)$. Then, the following conditions are equivalent.*

(a) *$G : \operatorname{Hom}_\Lambda(M,N) \to \operatorname{Hom}_\Gamma(GM, GN)$ is an isomorphism.*

(b) *$\operatorname{Hom}_\Lambda(M, I \otimes N) = 0$.*

Proof. Consider the natural inclusion $i_1 : (M,N) \to (M,N) \oplus (M, I \otimes N)$. Then, by 4.1, we know that $\delta i_1 = G$ and thus the result follows since δ is an isomorphism. □

Theorem 4.3. *For any $M \in \operatorname{mod}(\Lambda)$, the algebra $\operatorname{End}_\Gamma(GM)$ is an split-by-nilpotent extension of $\operatorname{End}_\Lambda(M)$ by $\operatorname{Hom}_\Lambda(M, I \otimes M)$.*

Proof. Let $M \in \operatorname{mod}(\Lambda)$. By 4.1, we have the exact sequence of R-modules

$$0 \longrightarrow \operatorname{Hom}_\Lambda(M, I \otimes M) \xrightarrow{L} \operatorname{End}_\Gamma(GM) \xrightarrow{\vartheta} \operatorname{End}_\Lambda(M) \longrightarrow 0,$$

where $\vartheta := \pi_1\delta^{-1}$ and $\pi_1 : \mathrm{End}_\Lambda(M) \oplus \mathrm{Hom}_\Lambda(M, I \otimes M) \to \mathrm{End}_\Lambda(M)$ is the canonical projection. Observe that $G : \mathrm{End}_\Lambda(M) \to \mathrm{End}_\Gamma(GM)$ is a ring morphism and ϑG is the identity map.

We assert that $\vartheta : \mathrm{End}_\Gamma(GM) \to \mathrm{End}_\Lambda(M)$ is a ring morphism. Indeed, we firstly transfer, by using the isomorphism δ, the multiplicative structure of the ring $\mathrm{End}_\Gamma(GM)$ to the R-module $\mathrm{End}_\Lambda(M) \oplus \mathrm{Hom}_\Lambda(M, I \otimes M)$. That is, for any $\alpha = (f_\alpha, g_\alpha)$ and $\beta = (f_\beta, g_\beta)$ in $\mathrm{End}_\Lambda(M) \oplus \mathrm{Hom}_\Lambda(M, I \otimes M)$, we set $\alpha\beta := \delta^{-1}(\delta(\alpha)\delta(\beta))$. In what follows, we shall compute the above product and show that

$$(*) \qquad \delta(\alpha)\delta(\beta) = G(f_\alpha f_\beta) + L(g_\alpha f_\beta + \varepsilon).$$

Indeed $\delta(\alpha)\delta(\beta) = G(f_\alpha)G(f_\beta) + L(g_\alpha)G(f_\beta) + L(g_\alpha)L(g_\beta) + G(f_\alpha)L(g_\beta)$. But $L(g_\alpha)G(f_\beta) = L(g_\alpha f_\beta)$ since $L(g_\alpha)G(f_\beta)(\gamma \otimes m) = L(g_\alpha)(\gamma \otimes f_\beta(m)) = \gamma g_\alpha(f_\beta(m)) = L(g_\alpha f_\beta)(\gamma \otimes m)$. To compute $\mu := L(g_\alpha)L(g_\beta) + G(f_\alpha)L(g_\beta)$, we proceed as follows. Observe that $\mu = (G(f_\alpha) + L(g_\alpha))L(g_\beta) = \delta(\alpha)L(g_\beta)$. Consider the morphism $(0, g_\beta) : M \to GM$. Using the fact that $I \trianglelefteq \Gamma$, it can be seen that $\mathrm{Im}\,(\delta(\alpha)(0, g_\beta)) \subseteq I \otimes M$. Thus, the morphism ε, which is the composition of $M \to \mathrm{Im}\,(\delta(\alpha)(0, g_\beta)) \subseteq I \otimes M$, satisfies that $L(\varepsilon) = \mu$. Indeed, $L(\varepsilon)(\gamma \otimes m) = \gamma\varepsilon(m) = \gamma\delta(\alpha)(0, g_\beta)(m) = \gamma\delta(\alpha)(g_\beta(m)) = \delta(\alpha)(\gamma g_\beta(m)) = \delta(\alpha)(L(g_\beta)(\gamma \otimes m)) = \delta(\alpha)L(g_\beta)(\gamma \otimes m)$; proving $(*)$. Now, we are ready to prove that $\vartheta : \mathrm{End}_\Gamma(GM) \to \mathrm{End}_\Lambda(M)$ is a ring homomorphism. That is, by $(*)$, we have that $\vartheta(\delta(\alpha)\delta(\beta)) = \pi_1\delta^{-1}(\delta(\alpha)\delta(\beta)) = f_\alpha f_\beta = \pi_1\delta^{-1}\delta(\alpha)\pi_1\delta^{-1}\delta(\beta) = \vartheta(\delta(\alpha))\vartheta(\delta(\beta))$.

Finally, we prove that $\mathrm{Im}\,(L) \subseteq \mathrm{rad}\,(\mathrm{End}_\Gamma(GM))$. To see that, it is enough to check that the ideal $\mathrm{Im}\,(L)$ is nilpotent. Let $g_1, g_2, \cdots, g_n \in \mathrm{Hom}_\Lambda(M, I \otimes M)$, $\gamma \in \Gamma$ and $m \in M$. Since $L(g_1)L(g_2)\cdots L(g_n)(\gamma \otimes m) \in I^n \otimes M$; and using the fact that I is nilpotent, it follows that $\mathrm{Im}\,(L)$ is also nilpotent. $\square$

5. Extending stratifying systems with the functor G

In this section, we consider a split-by-nilpotent extension Γ of Λ by I. As we have seen before, there is the change of rings functor $G := {}_\Gamma\Gamma_\Lambda \otimes_\Lambda - : \mathrm{mod}\,(\Lambda) \to \mathrm{mod}\,(\Gamma)$. We give conditions for the image under G, of a stratifying system in $\mathrm{mod}\,(\Lambda)$, to be a stratifying system in $\mathrm{mod}\,(\Gamma)$.

Definition 5.1. *A stratifying system* $(\Theta, \leq)$, *of size* t *in* $\mathrm{mod}\,(\Lambda)$, *is* **compatible** *with the ideal* $I \trianglelefteq \Gamma$ *if the following conditions hold.*

(C1) $\mathrm{Hom}_\Lambda(\Theta(j), I \otimes_\Lambda \Theta(i)) = 0$ *for* $j > i$.
(C2) $\mathrm{Ext}^1_\Lambda(\Theta(j), I \otimes_\Lambda \Theta(i)) = 0$ *for* $j \geq i$.

Proposition 5.2. *Let* $\Theta = \{\Theta(i)\}_{i=1}^t$ *be objects in* $\mathrm{mod}\,(\Lambda)$ *and* $\leq$ *be a linear order on* $[1, t]$. *If* I_Λ *is projective, then the following conditions are equivalent.*

(a) $(G(\Theta), \leq)$ *is a stratifying system in* $\mathrm{mod}\,(\Gamma)$.
(b) $(\Theta, \leq)$ *is a stratifying system in mod*(Λ), *which is compatible with the ideal* I.

Proof. By 3.1, we know that the functor $G : \mathrm{mod}\,(\Lambda) \to \mathrm{mod}\,(\Gamma)$ reflects and preserves indecomposables. On the other hand, since I_Λ is projective, we have by 3.2 that

$$\mathrm{Ext}^i_\Gamma(G(X), G(Y)) \simeq \mathrm{Ext}^i_\Lambda(X, Y) \oplus \mathrm{Ext}^i_\Lambda(X, I \otimes_\Lambda Y),$$

for all $X, Y \in \mathrm{mod}\,(\Lambda)$ and any i. Thus, the equivalence between (a) and (b) follows. □

Corollary 5.3. *Let $\Theta = \{\Theta(i)\}_{i=1}^t$ be objects in $\mathrm{mod}\,(\Lambda)$, and let $\leq$ be a linear order on $[1, t]$. If ${}_\Lambda I_\Lambda \in \mathrm{add}\,({}_\Lambda\Lambda_\Lambda)$ then the following conditions are equivalent.*

(a) *$(G(\Theta), \leq)$ is a stratifying system in $\mathrm{mod}\,(\Gamma)$.*
(b) *$(\Theta, \leq)$ is a stratifying system in $\mathrm{mod}\,(\Lambda)$.*

Proof. Let ${}_\Lambda I_\Lambda \in \mathrm{add}\,({}_\Lambda\Lambda_\Lambda)$. Then $I \otimes_\Lambda X \in \mathrm{add}\,(X)$ for any $X \in \mathrm{mod}\,(\Lambda)$. Therefore, any stratifying system in $\mathrm{mod}\,(\Lambda)$ is compatible with the ideal I. So, the result follows from 5.2. □

Let us consider the following examples.

Example 5.4. *Consider the trivial extension $\Gamma := \Lambda \ltimes I$ of an algebra Λ by its minimal injective cogenerator $I := D(\Lambda)$. If Λ is a symmetric algebra, it is well known that ${}_\Lambda I_\Lambda \simeq {}_\Lambda\Lambda_\Lambda$ as bimodules. Therefore, in this case, the needed hypothesis in 5.3 holds. We recall that in [12] stratifying systems for symmetric special biserial algebras are constructed.*

Example 5.5. *Let Q be the quiver $\bullet^3 \xrightarrow{\beta} \bullet^1 \xrightarrow{\alpha} \bullet^2$. Consider the quotient path k-algebra $\Gamma := kQ/\langle\alpha\beta\rangle$ with the ideal $I := \langle\overline{\beta}\rangle \trianglelefteq \Gamma$. Then Γ is a split-by-nilpotent extension of $\Lambda := \Gamma/I$ by I. Moreover, the ordinary quiver Q_Λ of Λ is $\bullet^3 \quad \bullet^1 \xrightarrow{\alpha} \bullet^2$. We consider the natural order $1 \leq 2 \leq 3$. Then, we have the canonical stratifying system $({}_\Lambda\Delta, \leq)$ in $\mathrm{mod}\,(\Lambda)$ where ${}_\Lambda\Delta(i) = {}_\Lambda S(i)$ is the simple Λ-module associated to the vertex $i \in Q_\Lambda$. It can be seen that $I_\Lambda \simeq e_3\Lambda$ (where e_3 is the primitive idempotent associated with the vertex 3), $I \otimes_\Lambda {}_\Lambda\Delta(1) = 0 = I \otimes_\Lambda {}_\Lambda\Delta(2)$ and $I \otimes_\Lambda {}_\Lambda\Delta(3) \simeq {}_\Lambda\Delta(1)$. Thus, the stratifying system $({}_\Lambda\Delta, \leq)$ is compatible with the ideal I; and so by 5.2 it follows that $(G({}_\Lambda\Delta), \leq)$ is a stratifying system in $\mathrm{mod}\,(\Gamma)$.*

As we have seen in 5.2, the notion of stratifying system compatible with the ideal I plays an important role. In the following proposition, we give conditions for the canonical stratifying system to be compatible with the ideal I. For doing so, let ${}_\Lambda P = \{{}_\Lambda P(i)\}_{i=1}^n$ be a representative set of the indecomposable projective Λ-modules, where $n := rk\, K_0(\Lambda)$, and let $\leq$ be a linear order on the set $[1, n] := \{1, 2, \cdots, n\}$. Let us consider the set of standard Λ-modules ${}_\Lambda\Delta$, computed by using the pair $({}_\Lambda P, \leq)$, and also the representative set ${}_\Lambda S = \{{}_\Lambda S(i)\}_{i=1}^n$ of simple Λ-modules, where ${}_\Lambda S(i) := {}_\Lambda P(i)/\mathrm{rad}\,({}_\Lambda P(i))$ for each i. Recall that each ${}_\Lambda\Delta(i)$ has composition factors

only amongst ${}_\Lambda S(j)$ with $j \leq i$. That is, the multiplicity $[{}_\Lambda\Delta(i) : {}_\Lambda S(j)]$ of the simple ${}_\Lambda S(j)$ in ${}_\Lambda\Delta(i)$ is equal to zero for $j > i$. So, we start with the following definition.

Definition 5.6. *The pair $({}_\Lambda S, \leq)$ is admissible with the ideal $I \trianglelefteq \Gamma$ if*

$$[I \otimes {}_\Lambda S(i) : {}_\Lambda S(j)] = 0 \quad \text{for} \quad j > i.$$

Lemma 5.7. *Let I_Λ be a projective Λ-module. Then, the following statements are equivalent.*

(a) *The pair $({}_\Lambda S, \leq)$ is admissible with the ideal I.*
(b) $[I \otimes {}_\Lambda\Delta(i) : {}_\Lambda S(j)] = 0$ *for $j > i$.*

Proof. (a) $\Rightarrow$ (b) It follows from the fact that $I \otimes_\Lambda - : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Lambda)$ is an exact functor and ${}_\Lambda\Delta(i) \in \mathfrak{F}(\{{}_\Lambda S(t) \ : \ t \leq i\})$ for each i.

(b) $\Rightarrow$ (a) Let $j > i$. By Applying the exact functor $I \otimes_\Lambda -$ to the exact sequence $0 \to \operatorname{rad}({}_\Lambda\Delta(i)) \to {}_\Lambda\Delta(i) \to {}_\Lambda S(i) \to 0$, we get the exact sequence $0 \to I \otimes \operatorname{rad}({}_\Lambda\Delta(i)) \to I \otimes {}_\Lambda\Delta(i) \to I \otimes {}_\Lambda S(i) \to 0$. Thus, the condition given in (b) implies that $[I \otimes {}_\Lambda S(i) : {}_\Lambda S(j)] = 0$; proving (a). $\square$

The following result relates the admissibility of $({}_\Lambda S, \leq)$ with the compatibility of $({}_\Lambda\Delta, \leq)$.

Proposition 5.8. *Let I_Λ be a projective Λ-module. Then, the following statements hold true.*

(a) *If $({}_\Lambda S, \leq)$ is admissible with the ideal I, then the canonical stratifying system $({}_\Lambda\Delta, \leq)$ is compatible with the ideal I.*
(b) *If Λ is an ss-algebra such that $({}_\Lambda\Delta, \leq)$ is compatible with the ideal I, then $({}_\Lambda S, \leq)$ is admissible with the ideal I.*

Proof. (a) $\Rightarrow$ (b) Assume that, for each $i \in [1, n]$, the Λ-module $I \otimes {}_\Lambda S(i)$ has composition factors only amongst ${}_\Lambda S(j)$ with $j \leq i$. Then, by 5.7, we get that $[I \otimes {}_\Lambda\Delta(j) : {}_\Lambda S(i)] = 0$ for $i > j$. Therefore $\operatorname{Hom}_\Lambda({}_\Lambda P(i), I \otimes {}_\Lambda\Delta(j)) = 0$ for $i > j$, and so the condition (C1) in 5.1 holds.

Let now $i \geq j$, and let $\nu \ : \ 0 \to I \otimes {}_\Lambda\Delta(j) \to U \xrightarrow{\alpha} {}_\Lambda\Delta(i) \to 0$ be an exact sequence. So, by 5.7 (b), we get that $U \in \mathfrak{F}(\{{}_\Lambda S(t) \ : \ t \leq i\})$. Consider the epimorphism $p : {}_\Lambda P(i) \to {}_\Lambda\Delta(i)$ where $\operatorname{Ker}(p) = \operatorname{Tr}_{\oplus_{r>i}\, {}_\Lambda P(r)}({}_\Lambda P(i))$. Then, there is a morphism $f : {}_\Lambda P(i) \to U$ such that $p = \alpha f$. By taking the factorization ${}_\Lambda P(i) \xrightarrow{\overline{f}} \operatorname{Im}(f) \xrightarrow{\imath} U$ of f throughout its image, we have that $(\alpha\imath)\overline{f} = p$. That is, the quotient morphism $\overline{f} : {}_\Lambda P(i) \to \operatorname{Im}(f)$ factors throughout $p : {}_\Lambda P(i) \to {}_\Lambda\Delta(i)$. Moreover, since $\operatorname{Ker}(p) = \operatorname{Tr}_{\oplus_{r>i}\, {}_\Lambda P(r)}({}_\Lambda P(i))$, it follows that p factors throughout $\overline{f}$. Hence $\alpha\imath : \operatorname{Im}(f) \to {}_\Lambda\Delta(i)$ is an isomorphism. Therefore, the exact sequence ν splits, and so the condition (C2) in 5.1 holds.

(b) $\Rightarrow$ (a) Assume that Λ is an ss-algebra such that $({}_\Lambda\Delta, \leq)$ is compatible with the ideal I. Let $j > i$ and consider the canonical exact sequence η : $0 \to K(j) \to {}_\Lambda P(j) \to {}_\Lambda\Delta(j) \to 0$, where $K(j) := \operatorname{Tr}_{\oplus_{r>j}\, {}_\Lambda P(r)}({}_\Lambda P(j))$.

Since Λ is an ss-algebra, it is known that $K(j) \in \mathfrak{F}(\{{}_\Lambda\Delta(t) \ : \ t > j\})$. Thus $\mathrm{Hom}_\Lambda(K(j), I \otimes {}_\Lambda\Delta(i)) = 0$ (see 5.1 (C1)). Applying $\mathrm{Hom}_\Lambda(-, I \otimes {}_\Lambda\Delta(i))$ to η, and since $({}_\Lambda\Delta, \leq)$ is compatible with the ideal I, we conclude that $\mathrm{Hom}_\Lambda({}_\Lambda P(j), I \otimes_\Lambda \Delta(i)) \simeq \mathrm{Hom}_\Lambda(K(j), I \otimes_\Lambda \Delta(i)) = 0$. Therefore $[I \otimes_\Lambda \Delta(i) : {}_\Lambda S(j)] = 0$ for $j > i$. Finally, by 5.7 (b), we conclude that $({}_\Lambda S, \leq)$ is admissible with the ideal I, $\square$

Let I_Λ be a projective Λ-module. In [25], the authors consider as a main hypothesis that the Λ-module $I \otimes_\Lambda S(i)$ has composition factors only amongst ${}_\Lambda S(j)$ with $j \leq i$. As we have seen in 5.8, under the hypothesis that Λ is an ss-algebra, this is equivalent to the compatibility condition (see 5.1) for the canonical stratifying system. Observe that 5.1 is precisely the needed condition to determine when a stratifying system in $\mathrm{mod}\,(\Lambda)$ can be extended, throughout the functor G, to a stratifying system in $\mathrm{mod}\,(\Gamma)$ (see 5.2).

Proposition 5.9. *Let I_Λ be a projective Λ-module and let $(\Theta, \leq)$ be a stratifying system of size t in $\mathrm{mod}\,(\Lambda)$, which is compatible with the ideal I. Then, $(G\Theta, \leq)$ is a stratifying system of size t in $\mathrm{mod}\,(\Gamma)$, and the following statements hold true.*

(a) *The restriction $F|_{\mathfrak{F}(G\Theta)} : \mathfrak{F}(G\Theta) \to \mathfrak{F}(\Theta)$ is well defined, and it is an exact, faithful and dense functor which reflects indecomposables.*
(b) *The restriction $G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is well defined, and it is an exact and faithful functor which preserves and reflects indecomposables.*
(c) *The restriction $F|_{\mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)})} : \mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)}) \to \mathfrak{F}(\Theta)$ is an equivalence of categories, and a quasi-inverse is the restriction $G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)})$.*

Proof. By 5.2 we know that $(G\Theta, \leq)$ is a stratifying in $\mathrm{mod}\,(\Gamma)$.

(a) Let $M \in \mathfrak{F}(G\Theta)$. We prove, by induction on the $G\Theta$-length $\ell_{G\Theta}(M)$, that $F(M) \in \mathfrak{F}(\Theta)$ and $\mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, M) = 0$. If $\ell_{G\Theta}(M) = 1$ then $M \simeq G\Theta(i)$ for some i. Thus $F(M) \simeq FG\Theta(i) \simeq \Theta(i)$ (see 3.1 (a)) and $\mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, M) \simeq \mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, G\Theta(i)) = 0$ (see 3.2 (a)).
Let $\ell_{G\Theta}(M) > 1$. Then, from [21, Lemma 2.8], there is an exact sequence $\eta \ : \ 0 \to G\Theta(i) \to M \to N \to 0$ in $\mathfrak{F}(G\Theta)$, with $\ell_{G\Theta}(N) = \ell_{G\Theta}(M) - 1$. Applying the functor F to η, we get the exact sequence $\mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, G\Theta(i)) \to \mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, M) \to \mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, N) \to FG\Theta(i) \to FM \to FN \to 0$. By induction we know that $\mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, N) = 0$ and $FN \in \mathfrak{F}(\Theta)$. Thus $F(M) \in \mathfrak{F}(\Theta)$ and $\mathrm{Tor}_1^\Gamma(\Lambda_\Gamma, M) = 0$. In particular, it follows that the restriction $F|_{\mathfrak{F}(G\Theta)}$ is well defined and it is also an exact functor. Moreover, by 3.1, it is also a faithful and dense functor which reflects indecomposables.

(b) Since Γ_Λ is projective (see 3.2 (a)), it follows that $G : \mathrm{mod}\,(\Lambda) \to \mathrm{mod}\,(\Gamma)$ is an exact functor. Hence the restriction $G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is

well defined. Finally, from 3.1 (b) and (c), we conclude that G is faithful and also preserves and reflects indecomposables.

(c) It follows from (a), (b) and 3.3 (a). $\square$

Corollary 5.10. *Let I_Λ be a projective Λ-module and let $(\Theta, \leq)$ be a stratifying system of size t in $\mathrm{mod}\,(\Lambda)$, which is compatible with the ideal I. Then, the following conditions are equivalent.*

(a) *The restriction functor $F|_{\mathfrak{F}(G\Theta)} : \mathfrak{F}(G\Theta) \to \mathfrak{F}(\Theta)$ is an exact equivalence of categories, and its quasi-inverse is the restriction functor $G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$.*

(b) $\mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)}) = \mathfrak{F}(G\Theta)$.

(c) *The class $\mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)})$ is closed under extensions in* $\mathrm{mod}\,(\Gamma)$.

Proof. It follows from 5.9 and the fact that $\mathfrak{F}(G\Theta)$ is the smaller full subcategory of $\mathrm{mod}\,(\Gamma)$, which contains $G\Theta$ and is closed under extensions. $\square$

Observe that, in general, the class $\mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)})$ is not necessarily closed under extensions in $\mathrm{mod}\,(\Gamma)$. A sufficient condition for the equality $\mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)}) = \mathfrak{F}(G\Theta)$ will be given in 5.15.

Proposition 5.11. *Let I_Λ be projective, $(\Theta, \leq)$ be a stratifying system of size t in $\mathrm{mod}\,(\Lambda)$, which is compatible with the ideal I, and let $(\Theta, \underline{Q}, \leq)$ be the Ext-projective stratifying system associated to $(\Theta, \leq)$. Then $\mathrm{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0$ if and only if the triple $(G\Theta, G\underline{Q}, \leq)$ is the Ext-projective stratifying system associated to the stratifying system $(G\Theta, \leq)$.*

Proof. We assert that $\mathrm{Ext}^1_\Gamma(GQ(i), G\Theta(j)) \simeq \mathrm{Ext}^1_\Lambda(Q(i), I \otimes \Theta(j))$ for any $i, j \in [1, t]$. Indeed, since $\mathrm{Ext}^1_\Lambda(Q(i), \Theta(j)) = 0$ for any i, j, then by 3.2 (b), the assertion follows. Thus, by the above assertion, the implication "$\Leftarrow$" is clear. Assuming that $\mathrm{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0$, we obtain from our assertion that GQ is Ext-projective in $G\Theta$. Furthermore, using the fact that $G|_{\mathfrak{F}(\Theta)}$ is an exact functor (see 5.9 (b)), we get that the fundamental sequence $\varepsilon_i \ : \ 0 \to K(i) \to Q(i) \to \Theta(i) \to 0$, attached to the system $(\Theta, \underline{Q}, \leq)$ (see 2.2 (b)) gives the fundamental sequence $G\varepsilon_i \ : \ 0 \to GK(i) \to G\overline{Q}(i) \to G\Theta(i) \to 0$ corresponding to the system $(G\Theta, G\underline{Q}, \leq)$. $\square$

Theorem 5.12. *Let I_Λ be projective, $(\Theta, \leq)$ be a stratifying system of size t in $\mathrm{mod}\,(\Lambda)$, which is compatible with the ideal I and $\mathrm{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0$, and let $(\Theta, \underline{Q}, \leq)$ be the Ext-projective stratifying system associated to $(\Theta, \leq)$. Consider the algebras $A := \mathrm{End}_\Lambda(Q)^{op}$ and $GA := \mathrm{End}_\Gamma(GQ)^{op}$, where $Q := \oplus_{i=1}^t Q(i)$. Then, the following statements hold true.*

(a) *GA is an split-by-nilpotent extension of A by $\mathrm{Hom}_\Lambda(Q, I \otimes Q)$. Furthermore, both algebras A and $G(A)$ are basic, standardly stratified and $rk\, K_0(A) = t = rk\, K_0(GA)$.*

(b) $\overline{G} := \mathrm{Hom}_\Gamma(GQ,-) \circ G|_{\mathfrak{F}(\Theta)} \circ Q \otimes_A - : \mathfrak{F}({}_A\Delta) \to \mathfrak{F}({}_{GA}\Delta)$ *is well defined and it is an exact and faithful functor which preserves and reflects indecomposables, and* $\overline{G}({}_A\Delta(i)) \simeq {}_{GA}\Delta(i)$ *for any* $i \in [1,t]$.

(c) $\overline{F} := \mathrm{Hom}_\Lambda(Q,-) \circ F|_{\mathfrak{F}(G\Theta)} \circ GQ \otimes_{GA} - : \mathfrak{F}({}_{GA}\Delta) \to \mathfrak{F}({}_A\Delta)$ *is well defined and it is an exact, dense and faithful functor which reflects indecomposables, and* $\overline{F}({}_{GA}\Delta(i)) \simeq {}_A\Delta(i)$ *for any* $i \in [1,t]$.

(d) *A is quasi-hereditary if and only if GA is quasi-hereditary.*

(e) *The restriction functor* $\overline{F}|_{\mathrm{proj}\,(GA)} : \mathrm{proj}\,(GA) \to \mathrm{proj}\,(A)$ *is an equivalence and its quasi-inverse is* $\overline{G}|_{\mathrm{proj}\,(A)} : \mathrm{proj}\,(A) \to \mathrm{proj}\,(GA)$.

Proof. (a) It follows from 4.3 and [21, Theorem 3.2 (a)].

(b) By [21, Theorem 3.2], it follows that the functors $Q \otimes_A - : \mathfrak{F}({}_A\Delta) \to \mathfrak{F}(\Theta)$ and $\mathrm{Hom}_\Gamma(GQ,-) : \mathfrak{F}(G\Theta) \to \mathfrak{F}({}_{GA}\Delta)$ are exact equivalences. Thus, by 5.9 (b), the functor $\overline{G} := \mathrm{Hom}_\Gamma(GQ,-) \circ G|_{\mathfrak{F}(\Theta)} \circ Q \otimes_A - : \mathfrak{F}({}_A\Delta) \to \mathfrak{F}({}_{GA}\Delta)$ is well defined and has the desired properties. Moreover, by [21, Theorem 3.1], we have that $Q_A \otimes {}_A\Delta(i) \simeq \Theta(i)$ and $\mathrm{Hom}_\Gamma(GQ, GQ(i)) \simeq {}_{GA}\Delta(i)$. Therefore $\overline{G}({}_A\Delta(i)) \simeq {}_{GA}\Delta(i)$ for any $i \in [1,t]$.

(c) It follows, as in the proof of (b), from [21, Theorem 3.2] and 5.9 (a).

(d) By [21, Theorem 3.2], it is enough to see that: $\mathrm{rad}\,(\mathrm{End}_\Lambda(\Theta(i))) = 0$ if and only if $\mathrm{rad}\,(\mathrm{End}_\Gamma(G\Theta(i))) = 0$ for any $i \in [1,t]$. By 5.9 (a) and (b), we have the ring morphisms $\mathrm{End}_\Lambda(\Theta(i)) \xrightarrow{G} \mathrm{End}_\Gamma(G\Theta(i)) \xrightarrow{F} \mathrm{End}_\Lambda(\Theta(i))$. Thus $G(\mathrm{rad}\,(\mathrm{End}_\Lambda(\Theta(i)))) \subseteq \mathrm{rad}\,(\mathrm{End}_\Gamma(G\Theta(i)))$ and also we have the inclusion $F(\mathrm{rad}\,(\mathrm{End}_\Gamma(G\Theta(i)))) \subseteq \mathrm{rad}\,(\mathrm{End}_\Lambda(\Theta(i)))$. Therefore, the desired equivalence holds by the fact that F and G are faithful functors.

(e) It follows from 3.3 (b) and [21, Theorem 3.2]. $\square$

Remark 5.13. *Let* $(\Theta, \underline{Q}, \leq)$ *be an Ext-projective stratifying system in* $\mathrm{mod}\,(\Lambda)$. *Observe that, if* $I \otimes \mathbf{\Theta} \in \mathfrak{F}(\Theta)$ *then* $\mathrm{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0$.

As we have seen above (see 5.9 and 5.10) the restriction functor $G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is not, in general, an equivalence. So, in the following results, we give sufficient conditions ensuring that $G|_{\mathfrak{F}(\Theta)}$ is an equivalence of categories. In order to do that, we start with the following lemma.

Lemma 5.14. *Let* $(\Theta, \underline{Q}, \leq)$ *be an Ext-projective stratifying system of size t, in* $\mathrm{mod}\,(\Lambda)$, *such that* $\mathrm{Hom}_\Lambda(\mathbf{\Theta}, I \otimes \mathbf{\Theta}) = 0$. *Then, the following statements hold true.*

(a) $\mathrm{Hom}_\Lambda(M, I \otimes N) = 0$ *for any* $M, N \in \mathfrak{F}(\Theta)$.

(b) $\mathrm{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0 \quad \Leftrightarrow \quad \mathrm{Ext}^1_\Lambda(\mathbf{\Theta}, I \otimes \mathbf{\Theta}) = 0$.

Proof. (a) It follows from [21, Lemma 2.8] and by induction on the Θ-length $\ell_\Theta(M)$ of N.

(b) The implication "$\Leftarrow$" follows easily since $Q \in \mathfrak{F}(\Theta)$. Let $\mathrm{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0$. Then, by applying $\mathrm{Hom}_\Lambda(-, I \otimes \Theta(j))$ to the canonical exact sequence $0 \to K(i) \to Q(i) \to \Theta(i) \to 0$ in $\mathfrak{F}(\Theta)$, we get that $\mathrm{Ext}^1_\Lambda(\Theta(i), I \otimes$

$\Theta(j)) = 0$ since $\mathrm{Hom}_\Lambda(K(i), I\otimes\Theta(j)) = 0 = \mathrm{Ext}^1_\Lambda(Q(i), I\otimes\Theta(j))$; and hence $\mathrm{Ext}^1_\Lambda(\boldsymbol{\Theta}, I\otimes\boldsymbol{\Theta}) = 0$. □

Theorem 5.15. *Let I_Λ be projective, $(\Theta, \underline{Q}, \leq)$ be an Ext-projective stratifying system of size t, in $\mathrm{mod}\,(\Lambda)$, such that $\mathrm{Hom}_\Lambda(\boldsymbol{\Theta}, I\otimes\boldsymbol{\Theta}) = 0 = \mathrm{Ext}^1_\Lambda(Q, I\otimes\boldsymbol{\Theta})$. Then, the following statements hold true.*

(a) *The stratifying system $(\Theta, \leq)$ is compatible with I, and $(G\Theta, G\underline{Q}, \leq)$ is the Ext-projective stratifying system associated to $(G\Theta, \leq)$.*

(b) *$G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is an exact equivalence of categories and $F|_{\mathfrak{F}(G\Theta)} : \mathfrak{F}(G\Theta) \to \mathfrak{F}(\Theta)$ is its quasi-inverse.*

Proof. By 5.14 (a) we have that $\mathrm{Hom}_\Lambda(M, I\otimes N) = 0$ for any $M, N \in \mathfrak{F}(\Theta)$. Thus, by 4.2, we conclude that $G = G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is a fully faithful functor. Furthermore, from 5.14 (b), it follows that $\mathrm{Ext}^1_\Lambda(\boldsymbol{\Theta}, I\otimes\boldsymbol{\Theta}) = 0$. In particular, the stratifying system $(\Theta, \leq)$ is compatible with I. Moreover, from 5.11, we conclude that $(G\Theta, G\underline{Q}, \leq)$ is the Ext-projective stratifying system associated to $(G\Theta, \leq)$; and hence (a) follows.

In order to prove (b), it is enough to see that $G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is dense. Indeed, if the restriction $G|_{\mathfrak{F}(\Theta)}$ is a dense functor, we would have that $\mathrm{Im}\,(G|_{\mathfrak{F}(\Theta)}) = \mathfrak{F}(G\Theta)$; and so from 5.10 we conclude (b).

Finally, we prove that the functor $G = G|_{\mathfrak{F}(\Theta)} : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is dense. Indeed, let $M \in \mathfrak{F}(G\Theta)$. We proceed by induction on the $G\Theta$-length $\ell_{G\Theta}(M)$. If $\ell_{G\Theta}(M) = 1$ then $M \simeq G\Theta(i)$ for some i.

Let $\ell_{G\Theta}(M) > 1$. Then, by [21, Lemma 2.8], there is an exact sequence $0 \to G\Theta(i) \to M \to M/G\Theta(i) \to 0$ in $\mathrm{mod}\,(\Gamma)$, where $\ell_{G\Theta}(M/G\Theta(i)) = \ell_{G\Theta}(M) - 1$ for some i. So, by induction, there exists $Z \in \mathfrak{F}(\Theta)$ such that $G(Z) \simeq M/G\Theta(i)$. Moreover, by [21, Proposition 2.10], there is an exact sequence $\eta_Z\ :\ 0 \to Z' \xrightarrow{u} Q_0(Z) \xrightarrow{\varepsilon_Z} Z \to 0$ in $\mathfrak{F}(\Theta)$, with $Q_0(Z) \in \mathrm{add}\,(Q)$. Thus, we get the following exact and commutative diagram in $\mathrm{mod}\,(\Gamma)$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G(Z') \arrow[r, equal] \arrow[d, "\mu"] & G(Z') \arrow[d, "G(u)"] & \\
\eta\ :\ 0 \arrow[r] & G\Theta(i) \arrow[r, "i_1"] \arrow[d, equal] & C \arrow[r, "p_2"] \arrow[d, "\lambda"] & G(Q_0(Z)) \arrow[r] \arrow[d, "G(\varepsilon_Z)"] & 0 \\
0 \arrow[r] & G\Theta(i) \arrow[r] & M \arrow[r] \arrow[d] & G(Z) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $G(Q_0(Z))$ is Ext-projective in $\mathfrak{F}(G\Theta)$, the exact sequence η splits and hence $C = G\Theta(i)\bigoplus G(Q_0(Z)) \simeq G(\Theta(i)\bigoplus G(Z))$, $i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $p_2 = (0,1)$. That is $\mu = \begin{pmatrix} \varphi \\ G(u) \end{pmatrix}$ with $\varphi : G(Z') \to G(\Theta(i))$. Using that the restriction $G|_{\mathfrak{F}(\Theta)}$ is full, there exists $h : Z' \to \Theta(i)$ such that $G(h) = \varphi$ and hence $\mu = G(\psi)$, where $\psi := \begin{pmatrix} h \\ u \end{pmatrix}$. Observe that the morphism ψ is a monomorphism since u is so. Then, by completing ψ to an exact sequence, we get the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Theta(i) \arrow[r, equal] \arrow[d] & \Theta(i) \arrow[d] & \\
0 \arrow[r] & Z' \arrow[r, "\psi"] \arrow[d, equal] & \Theta(i)\bigoplus Q_0(Z) \arrow[r] \arrow[d, "\pi_2"] & X \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & Z' \arrow[r, "u"] & Q_0(Z) \arrow[r, "\varepsilon_Z"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0, &
\end{tikzcd}$$

where the rows and columns are exact sequences and π_2 is the canonical projection. Observe that $X \in \mathfrak{F}(\Theta)$ since $\mathfrak{F}(\Theta)$ is closed under extensions. Thus, we get the exact sequence

$$0 \longrightarrow G(Z') \xrightarrow{G(\psi)} G(\Theta(i)\bigoplus Q_0(Z)) \longrightarrow G(X) \longrightarrow 0.$$

But $G(\psi) = \mu$ and hence $G(X) \simeq \mathrm{Coker}\,(\mu) = M$; proving that $G : \mathfrak{F}(\Theta) \to \mathfrak{F}(G\Theta)$ is dense. $\Box$

Let us consider the following examples.

Example 5.16. *Let Γ be the quotient path k-algebra given by the quiver*

$$\begin{tikzcd}
\bullet^1 \arrow[dd, "\sigma"'] \arrow[dr, "\alpha"] & & \bullet^6 \\
& \bullet^3 \arrow[ur, "\delta"] \arrow[dr, "\varepsilon"] & \\
\bullet^2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & \bullet^5 \\
& \bullet^4 \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

modulo the relations $\delta\beta = 0$ and $\varepsilon\beta = \lambda\gamma$. Consider the ideal $I = \langle \overline{\beta}, \overline{\gamma} \rangle \trianglelefteq \Gamma$. So, Γ is an split-by-nilpotent extension of $\Lambda := \Gamma/I$ by I. Furthermore, the algebra Λ is the path k-algebra given by the quiver

$$\begin{tikzcd}
& & \bullet^6 & & \\
\bullet^2 & \bullet^1 \arrow[l, "\sigma"'] \arrow[r, "\alpha"] & \bullet^3 \arrow[u, "\delta"] \arrow[r, "\varepsilon"] & \bullet^5 & \bullet^4 \arrow[l, "\lambda"']
\end{tikzcd}$$

Observe that I_Λ is projective, since $I_\Lambda \simeq {}_{\Lambda^{op}}P(2)^3$. We consider the natural order $1 \leq 2 \leq 3 \leq 4$ and the stratifying system $(\Theta, \leq)$ of size 4 in $\operatorname{mod}(\Lambda)$, where $\Theta(1) = \begin{smallmatrix} S(1) \\ S(3) \end{smallmatrix} = {}_\Lambda I(3)$, $\Theta(2) = \begin{smallmatrix} S(3)\ S(4) \\ S(5) \end{smallmatrix}$, $\Theta(3) = S(4)$, $\Theta(4) = S(2) = {}_\Lambda P(2)$. An explicit calculation gives us that $I \otimes \Theta(i) = 0$ for $i = 1, 2, 3$ and $I \otimes \Theta(4) \simeq \Theta(2)$. So $I \otimes \mathbf{\Theta} \in \mathfrak{F}(\Theta)$ and by 5.13 it follows that $\operatorname{Ext}^1_\Lambda(Q, I \otimes \mathbf{\Theta}) = 0$. Moreover, it can be seen that the stratifying system $(\Theta, \leq)$ is compatible with the ideal I. Thus, the needed conditions in 5.12 hold. Finally, observe that $\operatorname{Hom}_\Lambda(\Theta, I \otimes \mathbf{\Theta}) \neq 0$ since $\operatorname{Hom}_\Lambda(\Theta(2), I \otimes \Theta(4)) \simeq \operatorname{End}_\Lambda(\Theta(2)) \neq 0$.

Example 5.17. *Let Γ be the quotient path k-algebra given by the quiver*

$$\begin{tikzcd}
\bullet^1 \arrow[dr, "\alpha"] & & \bullet^6 \\
& \bullet^3 \arrow[ur, "\delta"] \arrow[dr, "\varepsilon"] & \\
\bullet^2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & \bullet^5 \\
& \bullet^4 \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

modulo the relations $\delta\beta = 0$ *and* $\varepsilon\beta = \lambda\gamma$. *Consider the ideal* $I = \langle \overline{\beta}, \overline{\gamma} \rangle \trianglelefteq \Gamma$. *So,* Γ *is an split-by-nilpotent extension of* $\Lambda := \Gamma/I$ *by* I. *Furthermore, the algebra* Λ *is the path* k*-algebra given by the quiver*

$$\begin{tikzcd}
 & & \bullet^{6} & & \\
\bullet^{2} & \bullet^{1} \arrow[r, "\alpha"] & \bullet^{3} \arrow[u, "\delta"] \arrow[r, "\varepsilon"] & \bullet^{5} & \bullet^{4} \arrow[l, "\lambda"']
\end{tikzcd}$$

Observe that I_Λ *is projective, since* $I_\Lambda \simeq {}_{\Lambda^{op}}P(2)^3$. *We consider the natural order* $1 \leq 2 \leq 3$ *and the stratifying system* $(\Theta, \leq)$ *of size* 3 *in* $\operatorname{mod}(\Lambda)$, *where* $\Theta(1) = \begin{array}{c} S(1) \\ S(4)\ S(3) \\ S(5) \end{array} = {}_\Lambda I(5)$, $\Theta(2) = S(2) = {}_\Lambda P(2)$ *and* $\Theta(3) = S(6) = {}_\Lambda P(6)$. *An explicit calculation gives us that* $I \otimes \Theta(i) = 0$ *for* $i = 1, 3$ *and* $N := I \otimes \Theta(2) = \begin{array}{c} S(3)\ S(4) \\ S(5) \end{array}$. *It can be seen that* $\operatorname{Hom}_\Lambda(\mathbf{\Theta}, I \otimes \mathbf{\Theta}) = 0 = \operatorname{Ext}^1_\Lambda(\mathbf{\Theta}, I \otimes \mathbf{\Theta})$. *Thus, by 5.14, the needed conditions in 5.15 hold. Finally, observe that* $I \otimes \mathbf{\Theta} \notin \mathfrak{F}(\Theta)$ *since* $I \otimes \Theta(2) = N \notin \mathfrak{F}(\Theta)$.

We finish this section by taking into consideration the canonical stratifying system $({}_\Lambda\Delta, \leq)$. Let the standard Λ-modules ${}_\Lambda\Delta$ be computed using the pair $({}_\Lambda P, \leq)$, where ${}_\Lambda P = \{{}_\Lambda P(i)\}_{i=1}^n$ is a representative set of the indecomposable projective Λ-modules, $n := rk\, K_0(\Lambda)$ and $\leq$ is a linear order on the set $[1, n]$. By 3.4, we have that ${}_\Gamma P := G({}_\Lambda P)$ is a representative set of the indecomposable projective Γ-modules. So, we compute the standard Γ-modules ${}_\Gamma\Delta$ by using the pair $({}_\Gamma P, \leq)$.

Theorem 5.18. *Let* I_Λ *be projective, and let* $({}_\Lambda\Delta, \leq)$ *be compatible with the ideal* I. *Then, the following statements hold true.*

(a) $G({}_\Lambda\Delta(i)) \simeq {}_\Gamma\Delta(i)$ *for any* $i \in [1, n]$.

(b) $G|_{\mathfrak{F}({}_\Lambda\Delta)} : \mathfrak{F}({}_\Lambda\Delta) \to \mathfrak{F}({}_\Gamma\Delta)$ *is an exact and faithful functor which preserves and reflects indecomposables.*

(c) $F|_{\mathfrak{F}({}_\Gamma\Delta)} : \mathfrak{F}({}_\Gamma\Delta) \to \mathfrak{F}({}_\Lambda\Delta)$ *is an exact, faithful and dense functor which reflects indecomposables.*

Proof. By 5.9, we have the stratifying system $(G({}_\Lambda\Delta), \leq)$ in $\operatorname{mod}(\Gamma)$ and also the exact functors $\mathfrak{F}({}_\Lambda\Delta) \xrightarrow{G} \mathfrak{F}(G({}_\Lambda\Delta)) \xrightarrow{F} \mathfrak{F}({}_\Lambda\Delta)$ satisfying the desired properties as in (b) and (c). It remains to show that $G({}_\Lambda\Delta(i)) \simeq {}_\Gamma\Delta(i)$ for any $i \in [1, n]$. Indeed, for each $i \in [1, n]$, consider the exact sequence

$$\eta_i \;:\; 0 \longrightarrow K(i) \longrightarrow {}_\Lambda P(i) \xrightarrow{p_i} {}_\Lambda\Delta(i) \longrightarrow 0,$$

where $K(i) := \operatorname{Tr}_{\oplus_{j>i}\, {}_\Lambda P(j)}\,({}_\Lambda P(i))$. Thus, we get the exact sequence

$$G(\eta_i) \;:\; 0 \longrightarrow G(K(i)) \longrightarrow {}_\Gamma P(i) \xrightarrow{G(p_i)} G({}_\Lambda\Delta(i)) \longrightarrow 0.$$

Let $Z(i) := \mathrm{Tr}_{\oplus_{j>i}\,{}_\Gamma P(j)}\,({}_\Gamma P(i))$, and let $\alpha : {}_\Gamma P(j) \to {}_\Gamma P(i)$, with $j > i$ be any morphism. Then, from 3.4, there exists $\alpha' : {}_\Lambda P(j) \to {}_\Lambda P(i)$ such that $G(\alpha') = \alpha$; and hence $G(p_i)\alpha = G(p_i\alpha') = 0$. Therefore $\mathrm{Im}\,(\alpha) \subseteq G(K(i))$; proving that $Z(i) \subseteq G(K(i))$. On the other hand, since $K(i) := \mathrm{Tr}_{\oplus_{j>i}\,{}_\Lambda P(j)}\,({}_\Lambda P(i))$ and G is an exact functor, we get an epimorphism $\oplus_{j>i}\,{}_\Gamma P(j)^m \to G(K(i))$, getting us that $G(K(i)) \subseteq Z(i)$. $\Box$

Corollary 5.19. *Let I_Λ be projective, and let $({}_\Lambda\Delta, \leq)$ be compatible with the ideal I. Then the following statements hold true.*

(a) *Λ is an standardly stratified (respectively, quasi-hereditary) algebra if and only if Γ is so.*

(b) *Let Λ be an standardly stratified algebra such that* $\mathrm{Hom}_\Lambda({}_{\boldsymbol\Lambda}\boldsymbol{\Delta}, I \otimes {}_{\boldsymbol\Lambda}\boldsymbol{\Delta}) = 0$. *Then, the functor $G|_{\mathfrak{F}({}_\Lambda\Delta)} : \mathfrak{F}({}_\Lambda\Delta) \to \mathfrak{F}({}_\Gamma\Delta)$ is an exact equivalence with quasi-inverse $F|_{\mathfrak{F}({}_\Gamma\Delta)} : \mathfrak{F}({}_\Gamma\Delta) \to \mathfrak{F}({}_\Lambda\Delta)$.*

Proof. (a) Let Λ be a standardly stratified algebra. For each $i \in [1, n]$, consider the exact sequence

$$\eta_i \;:\; 0 \longrightarrow K(i) \longrightarrow {}_\Lambda P(i) \xrightarrow{p_i} {}_\Lambda\Delta(i) \longrightarrow 0,$$

where $K(i) := \mathrm{Tr}_{\oplus_{j>i}\,{}_\Lambda P(j)}\,({}_\Lambda P(i))$. Since Λ is an ss-algebra, we have that η_i lies in $\mathfrak{F}({}_\Lambda\Delta)$. Thus, by 5.18 (a), we conclude that the exact sequence $G(\eta_i)$ lies in $\mathfrak{F}({}_\Gamma\Delta)$; proving that $\mathrm{proj}\,(\Gamma) \subseteq \mathfrak{F}({}_\Gamma\Delta)$. That is, the algebra Γ is standardly stratified.

Assume now that Γ is standardly stratified. Then, by [4], it follows that $\mathfrak{F}({}_\Gamma\Delta)$ is a resolving category. In particular, the exact sequence $G(\eta_i)$ lies in $\mathfrak{F}({}_\Gamma\Delta)$. Since $F|_{\mathfrak{F}({}_\Gamma\Delta)} : \mathfrak{F}({}_\Gamma\Delta) \to \mathfrak{F}({}_\Lambda\Delta)$ is exact, by applying F to $G(\eta_i)$, we get that the exact sequence η_i lies in $\mathfrak{F}({}_\Lambda\Delta)$ (see 3.1 (a)); proving that $\mathrm{proj}\,(\Lambda) \subseteq \mathfrak{F}({}_\Lambda\Delta)$. That is, the algebra Λ is standardly stratified.

The proof that Λ is quasi-hereditary if and only if Γ is so, can be done as we did in the proof of 5.12 (d). To do so, just replace there Θ by ${}_\Lambda\Delta$ and use 5.18 (a).

(b) Let Λ be a standardly stratified algebra. Then, the triple $({}_\Lambda\Delta, {}_\Lambda P, \leq)$ is the Ext-projective stratifying system associated to $({}_\Lambda\Delta, \leq)$. So the result follows from 5.15. $\Box$

6. Restricting stratifying systems with the functor F

In this section, let Γ be a split-by-nilpotent extension of Λ by and ideal $I \trianglelefteq \Gamma$. We consider the change of rings functor $F := {}_\Lambda\Lambda_\Gamma \otimes_\Gamma - : \mathrm{mod}\,(\Gamma) \to \mathrm{mod}\,(\Lambda)$. We ask under which conditions a stratifying system in $\mathrm{mod}\,(\Gamma)$ can be restricted, through the functor F, to a stratifying system in $\mathrm{mod}\,(\Lambda)$.

Observe that, by 3.1 (d), we know that F reflects indecomposables. However, the functor F, in general, could not preserve indecomposables. A sufficient condition, for the functor F, to preserve indecomposability is given in the following proposition. In order to do that, we will need the next lemma.

Lemma 6.1. *Let Λ_Γ be projective. Then, for any $M, N \in \operatorname{mod}(\Gamma)$, we have the following long exact sequence of R-modules*

$$0 \to \operatorname{Hom}_\Gamma(M, I \otimes_\Gamma N) \to \operatorname{Hom}_\Gamma(M, N) \to \operatorname{Hom}_\Lambda(F(M), F(N)) \to$$

$$\operatorname{Ext}^1_\Gamma(M, I \otimes_\Gamma N) \to \operatorname{Ext}^1_\Gamma(M, N) \to \operatorname{Ext}^1_\Lambda(F(M), F(N)) \to \operatorname{Ext}^2_\Gamma(M, I \otimes_\Gamma N).$$

Proof. Let $M, N \in \operatorname{mod}(\Gamma)$. We assert that

$$\operatorname{Ext}^i_\Gamma(M, \Lambda \otimes_\Gamma N) \simeq \operatorname{Ext}^i_\Lambda(F(M), F(N)) \text{ for any } i.$$

Indeed, we get that $\operatorname{Ext}^i_\Lambda(F(M), F(N)) \simeq \operatorname{Ext}^i_\Gamma(M, \operatorname{Hom}_\Lambda({}_\Lambda\Lambda_\Gamma, F(N)))$ since Λ_Γ is projective (see [30, Exercise 9.21]). On the other hand, from 2.7, we have that $\operatorname{Hom}_\Lambda({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma \otimes_\Gamma N) \simeq \operatorname{Hom}_\Lambda({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma) \otimes_\Gamma N \simeq {}_\Gamma\Lambda_\Gamma \otimes_\Gamma N$ as Γ modules; and so the assertion follows.

Applying the functor $- \otimes_\Gamma N$ to the exact sequence of $\Gamma - \Gamma$ bimodules $0 \to I \to \Gamma \xrightarrow{\pi} \Lambda \to 0$, and using the fact that $\operatorname{Tor}^\Gamma_1(\Lambda_\Gamma, {}_\Gamma N) = 0$, we get the exact sequence of Γ-modules $\eta \ : \ 0 \to I \otimes_\Gamma N \to N \to \Lambda \otimes_\Gamma N \to 0$. Moreover, by applying the functor $\operatorname{Hom}_\Gamma(M, -)$ to η, and the above assertion, we get the desired exact sequence. $\Box$

Proposition 6.2. *Let Λ_Γ be projective and $M \in \operatorname{mod}(\Gamma)$ be indecomposable. If $\operatorname{Hom}_\Gamma(M, I \otimes_\Gamma M) = 0 = \operatorname{Ext}^1_\Gamma(M, I \otimes_\Gamma M)$ then $F(M)$ is indecomposable.*

Proof. By the assumed conditions and 6.1, we get an isomorphism $\xi : \operatorname{End}_\Gamma(M) \to \operatorname{End}_\Lambda(F(M))$ of R-modules. An explicit computation of the map ξ gives us that $\xi(f)(\lambda \otimes m) = \lambda \otimes f(m)$ for any $\lambda \in \Lambda$, $m \in M$ and $f \in \operatorname{End}_\Gamma(M)$. Thus, the map ξ is also a ring homomorphism and hence $\operatorname{End}_\Gamma(M) \simeq \operatorname{End}_\Lambda(F(M))$ as R-algebras; proving that $F(M)$ is indecomposable. $\Box$

Definition 6.3. *A stratifying system $(\Psi, \leq)$, of size t in $\operatorname{mod}(\Gamma)$, is* **compatible** *with the ideal $I \trianglelefteq \Gamma$ if the following conditions hold.*

(C1) $\operatorname{Ext}^1_\Gamma(\Psi(j), I \otimes_\Gamma \Psi(i)) = 0$ *for $j > i$.*

(C2) $\operatorname{Ext}^2_\Gamma(\Psi(j), I \otimes_\Gamma \Psi(i)) = 0$ *for $j \geq i$.*

Theorem 6.4. *Let Λ_Γ be projective, and let $(\Psi, \leq)$ be a stratifying system of size t in $\operatorname{mod}(\Gamma)$, which is compatible with the ideal $I \trianglelefteq \Gamma$. Then, for each $i \in [1, t]$ and any choice of an indecomposable direct summand $\Theta(i)$ of $F(\Psi(i))$, the pair $(\Theta, \leq)$ is a stratifying system of size t in $\operatorname{mod}(\Lambda)$.*

Proof. Let $i \in [1, t]$. Since $\Psi(i) \neq 0$, it follows from 3.1 (b) that $F(\Psi(i)) \neq 0$. Thus $F(\Psi(i))$ has at least one indecomposable direct summand. Let $\Theta(i)$ be a choice of an indecomposable direct summand of $F(\Psi(i))$. Since $(\Psi, \leq)$ is compatible with the ideal I, we get from 6.1 that the following conditions hold: (a) $\operatorname{Hom}_\Lambda(F(\Psi(j)), F(\Psi(i))) = 0$ for $j > i$, and (b) $\operatorname{Ext}^1_\Lambda(F(\Psi(j)), F(\Psi(i))) = 0$ for $j \geq i$. Therefore, the same conditions, as in (a) and (b), hold for $\Theta :=$

$\{\Theta(i)\}_{i=1}^{t}$ since each $\Theta(i)$ is an indecomposable direct summand of $F(\Psi(i))$. Hence the result follows. $\square$

Example 6.5. *Let Γ be the split-by-nilpotent extension of Λ by I, which is considered in 5.5, and take the natural order $1 \leq 2 \leq 3$. Consider the pair $(\Psi, \leq)$, where $\Psi(1) := {}_{\Gamma}S(2)$, $\Psi(2) := {}_{\Gamma}P(1)$ and $\Psi(3) := {}_{\Gamma}S(3)$. Observe that $(\Psi, \leq)$ is a stratifying system in $\operatorname{mod}(\Gamma)$. Furthermore, since $I \otimes_{\Gamma} \Psi(1) = 0 = I \otimes_{\Gamma} \Psi(2)$, $I \otimes_{\Gamma} \Psi(3) \simeq {}_{\Gamma}S(1)$ and $\operatorname{id}({}_{\Gamma}S(1)) \leq 1$, it can be seen that the pair $(\Psi, \leq)$ is compatible with the ideal I; and furthermore Λ_{Γ} is projective since $\Lambda_{\Gamma} \simeq {}_{\Gamma^{op}}P(3) \oplus {}_{\Gamma^{op}}P(2)$. Thus by 6.4 it follows that, for any choice of an indecomposable direct summand $\Theta(i)$ of $F(\Psi(i))$, the pair $(\Theta, \leq)$ is a stratifying system of size 3 in $\operatorname{mod}(\Lambda)$. Since $F(\Psi(1)) \simeq {}_{\Lambda}P(2)$, $F(\Psi(2)) \simeq {}_{\Lambda}P(1)$ and $F(\Psi(3)) \simeq {}_{\Lambda}P(3)$, we see that the restriction of $(\Psi, \leq)$ to $\operatorname{mod}(\Lambda)$, through the functor F, gives only one stratifying system.*

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